Mathematical Knowledge and its Relation to the Knowledge of Mathematics Teachers: Linked Traumas and Resonances of Identity

PIERRE BERDOT, CLAUDINE BLANCHARD-LAVILLE, ALAIN BRONNER

The work of Alain Bronner (1997a, b) on problems of the school teaching and learning of real numbers has helped us to see that secondary mathematics teachers can fall prey to a certain type of internal conflict with respect to their relationship to knowledge. In the ensuing study, we try to show that in fact certain of these teachers feel attacked at a very core level in their relationship with mathematical knowledge, which is a major constituent of their professional self.

Since the teachers started teaching, syllabus reform [1] has been such that certain key objects of mathematical knowledge, which were part of their education and training, have been made to disappear. This loss is felt more or less strongly by teachers according to the coping strategies they develop, to be able to come to terms with the loss, and the compensations derived from them.

In our view, the psychic effects of these losses and 'attacks' are far more important than they first seemed. For this reason, we have sought to understand the underlying processes at work and, at the present time, feel it is possible to speak of actual traumas. These are linked to the various syllabus reforms which have occurred during the past twenty years, to which the teachers have fallen victim.

Confronted with what could, in their view, be considered as 'institutional mistreatment', and responding to the conflicts this arouses in them, the teachers individually establish forms of coping which compensate more or less well for the psychic suffering induced by the mistreatment. We describe here several key scenarios seen from this point of view.

This study led us to consider what value there could be in talking of 'institutional trauma' and to connect it to elements of the theory of trauma in a psychoanalytic setting. We were then led to seek out the origin of these traumas in the successive crises that mathematical knowledge has had to confront in the course of history. In this way, we came to see it as linked traumas (*traumatismes en chaîne*), a sort of genealogy of traumas. In historical reality, the traumas came to life in mathematicians at the moment of the discovery of new knowledge which overthrew their mathematical world.

The teaching institution, which needs to adjust to these changes in referential knowledge, suffers a certain number of consequences. Depending on the era, more or less adequate responses are offered. Effects of these traumas get transmitted to the teachers and their mathematical studies, via the intermediary of their teacher trainers at the university. Then, while teaching, the teachers risk passing on the effects of these traumas indirectly to students and do so all the more strongly if the institution fails to provide sufficient means for their negotiation. In fact, the institution remains silent at the level of coping with these 'crises' of knowledge in the process of didactic transposition.

Bronner, in regard to certain historical periods, has gone as far as to suggest the idea of an ‘institutional didactic void’. In other words, he proposes that certain institutional responses leave teachers dangling, as if on a wire over the *void*, at certain delicate points of teaching. It would be possible to believe at first blush that the academic competence of the teachers, which is not in any doubt here, would allow them to cope. However, in our opinion, it is a long way from being enough and this study tries to show that there are traces of ‘traumas’ on the psychic level.

Many psychoanalytic works today show that psychic traumas hang on for several generations by means of what is called ‘inter-generational psychic transmission’ (Kaës, 1993; Tisseron, 1995). In the professional register, we thought it feasible to move this idea over to the setting of the transmission of the profession of teaching – in other words, to teacher education. The effects we are discussing do not form part of the cognitive register. However, we know that the psychic dimension is not taken into account at present in teacher training: it is scarcely done so even in analysing teachers’ professional practice and is practically never called upon for questions involving teacher knowledge.

Introduction

Our starting point for this work was the didactic doctoral study of Alain Bronner (1997a), based on the results of an enquiry by means of questionnaires and interviews with secondary mathematics teachers, dealing with their way of teaching square roots. [2] Having analysed the transcripts of those whom Bronner interviewed about ‘square roots’ and ‘real numbers’, he identified four ways that teachers position themselves in relation to the demands of the institution:

- a position of strict conformity, where the teacher follows the instructions of the institution to the letter;

- a position of openness toward new numbers, in which the teachers try to show that new mathematical objects, other than decimals which are already known to the pupils, have the status of numbers;

- a position of openness toward non-decimal numbers, in which the teachers show that these new numbers are not decimals;

- a position of openness toward irrational numbers (that the institution does not require be introduced).

For what follows, the following elements of Bronner’s scheme are sufficient.

(1) The positions described in this classification are forms of response by teachers to the ‘institutional void’ mentioned earlier.

(2) Notice that in the latter two positions the teachers are carrying out a certain transgression of the official syllabi which do not require them, during this period of study, to introduce the character of new numbers in relation to square roots, while in the two former positions they submit to official instructions.

In addition to this classification of the institutional relation to the knowledge of teachers, Bronner showed by means of somewhat deeper interviews that the ways some of them had to adjust their teaching practices triggered a state of inner conflict.

It was this conclusion that attracted the attention of the other two authors of this article, Blanchard-Laville and Berdot, to the extent that it emphasised a form of suffering of the teacher-subject, the underlying moving forces of which we wanted to understand. It seemed to us that taking the psychic processes at work into account could bring a complementary source of insight to the interpretation of the distress of these teachers.

In consequence, we decided to extend this didactic study by working as a trio, in a more clinical manner, and by returning to the teacher interviews on which it was based from a new viewpoint, one more centred on identifying internal conflicts in the interviewed teachers, and their ways of adjusting to these conflicts.

This new interview analysis was carried out in a co-disciplinary manner (Blanchard-Laville, 1999), in an attempt to articulate both its didactic and psychic dimensions. This joint work primarily afforded an opportunity to combine the notion of the relation to knowledge as promoted by the didactical anthropological approach of Yves Chevallard (1988-9) – in terms of individual and institutional relations to objects of knowledge – with a more clinical conception of the relation to knowledge, such as that proposed by the team from the Nanterre CREF [3] (Beillerot, Blanchard-Laville and Mosconi, 1996).

Those interviewed were volunteers, mathematics teachers aged between 40 and 55, teaching grade 9 or 10 classes. Given their age, these teachers had undergone the ‘modern mathematics’ reform, [4] either as pupils and/or students in the main, though a few had begun to teach under the aegis of the syllabus of that period. Bronner framed the interview as part of a research project on the teaching of square roots, whose goal was to locate difficulties in teachers’ didactic choices for teaching this notion. The interviews were semi-structured and carried out by the researcher himself.

We followed up by analysing four of these interviews which seemed able to provide fruitful material for a more clinical analysis, even though these interviews were carried out for didactic ends in a guided, semi-structured manner.

One form of institutional mistreatment: Jacques and Heron’s algorithm

First of all, we present one of the interviewed teachers, whom we call Jacques. He is fifty-five years old, teaches in a suburban [5] junior high school and has several grade nine classes. The semi-structured interview, carried out at his school, generated a certain number of themes around the teaching of square root in ninth grade. The interview was carried out prior to the corresponding teaching sequence intended to have the teacher explain the didactic choices he thought he should make for this future sequence, as well as difficulties with which he thought he would be confronted in implementing his choices.

As an aside, let us make clear that the teaching of square root in the current syllabus is primarily concerned with transmitting the rules of algebraic calculation with this object: the product or quotient rule, for example $\sqrt{(3 \times 5)} = \sqrt{3} \times \sqrt{5}$. On the other hand, numbertheoretic properties are not studied, i.e. the fact that the square roots of certain numbers – such as $\sqrt{2}$ – are 'irrational' is not supposed to be taught.

Yet, this prototypical example ("Show that $\sqrt{2}$ is an irrational number") was in fact a task in many textbooks a few years ago. This type of exercise has now disappeared or been relegated to the very end of the chapter on square roots. Moreover, irrational numbers as such are unknown to pupils up to grade 9 and the syllabus does not require the teachers to introduce them explicitly.

In addition, the non-decimal character of irrational square roots [6] is no longer to be introduced according to the official instructions either, even though decimals have played and continue to play a central role in numerical work presented to pupils during secondary schooling. Algorithms to calculate square roots are no longer taught either, as are still done for multiplication and division for instance, and most importantly of all, pupils are not taught to use the method of extracting square roots (a hand calculation method) which had been taught for centuries before disappearing during the 1970s. It is this non-negotiation of the passage from decimals to other numbers which creates a kind of 'didactic void'.

What struck us as emblematic of Jacques' discourse was his surprise at being asked questions about this object (square root) because, according to him, it is a question about which there would not be anything at all to say, or so very little. Square roots present no problems for him and pupils have no difficulties with this lesson.

> Honestly, for me, this lesson on square roots is a very simple one; and for the pupils too, to show them, have them understand, have them get through it. It's a lesson that is relatively accessible, in other words they receive it very well.

Note that this is how he starts the interview and that he repeats this comment in a recurring manner throughout his conversation. For instance, he says "Once again, I tell you it's a lesson that goes well" and he even ends the interview in this fashion:

> I will now ask you a question, why on earth are you studying square roots? For us, it's a lesson like any other.

He translates his surprise into an attempt to deflect the course of the interview and almost offers an alternative topic to think about.

> You should have asked me about Thales [7], which doesn't seem to me to be a lesson on a more subtle level, yet nevertheless pupils receive it much worse.

He tries to some extent to get rid of the theme proposed by the interviewer:

> the square root is just window dressing [...] for me, it's window dressing.

Jacques' insistence on blocking this line of questioning about the object of square root alerted us, all the more so since mathematics education research has clearly shown that introducing this object of knowledge in junior high school is highly problematic, to say the least. But nevertheless Jacques repeated several times:

> for me it's a lesson which doesn't run into major difficulties. [...] I think that 60-70% of the pupils achieve the average, which shows that it is not a difficult lesson.

However, when we look more closely at what he said, at what he pointed out as not being difficult, it is calculating techniques.

> As for techniques of handling them, calculations in the literal sense, they don't run into major problems. For instance, they understand really well how to simplify a radical. [8]

In other words, what seems easy for him or his pupils is roughly what the institution requires him to teach, as we saw before. On the other hand, as soon as he steps back, even just a little, from the strict directives of the syllabus, then he no longer has the same opinion:

> It's a lesson which goes well apart from the matter of radicals involving letters, but anyway that is not explicitly in the ninth grade syllabus.

These are the issues that appear on first reading, in clear response to the explicit objective of the interview. Nevertheless, his overly-vested insistence on the ease of this lesson and his desire to deflect the aim of the interview led us to examine his discourse beyond this more obvious aspect.

Behind this explicit discourse about its 'ease', we have perceived quite another discourse, allowing certain *regrets* to be heard. Jacques says, for instance:

> The use of calculators nowadays has done away with all calculations on square roots. Examples are not hard to find: when I was a pupil, I studied the method of extracting square roots. It's been tossed in the bin several years ago. Why? Because it was a very onerous method.

And he really expresses a form of nostalgia:

> It was, nevertheless, as important a way of thinking as that which leads to rationalising the denominator of a fraction involving a square root.

We draw attention to the fact that the regrets he expresses involve themes linked to the square root which are no longer in favour in the current syllabus, but which, bearing in mind his age, were centrally part of his own learning and were objects of his teaching right from the outset. As he himself emphasises:

> This learning, which was essential some years ago, is much less so now.

As for calculating with expressions containing radicals, which are outside the syllabus, he acknowledges:

> Those [exercises] I make them do them anyway, but only provided everything that precedes them has been well learnt. Everything else is quite classical and doesn't present any difficulty.

For us, the totality of his words bears witness to the fact that Jacques has some difficulty in renouncing certain ‘old’ mathematical objects (which have disappeared from the current syllabus or which have been brought back to other levels of teaching), in order to submit to the requirements of the institution. Thus, he allows himself to introduce notions and procedures which he, quite clearly, holds very dear but which result in him distancing himself from the official requirements.

> There, I tried a little algorithm [in fact, it is Heron’s algorithm [9]] which is not easy for a square root: in other words, to show that for every square root you can find a fraction whose value is very close to the square root. This algorithm is really good, but rather delicate, but if you do one or two examples, the pupils get it perfectly.

He tries to justify himself in the eyes of the interviewer, no doubt being aware of a sort of transgression of official instructions (we shall return a little later to the interview situation itself which, we believe, revives for Jacques his relation to the institution).

> I do it, because I find it interesting, it makes them work on quotients, on square roots, because they must find the integer part of the square root, etc., which makes them complete an extra exercise.

In fact, we have the impression that it is primarily for himself that he introduces this algorithm. It is as if this gesture allows him to recall his connection to mathematical knowledge, by linking it to fundamental objects, which he encountered in his studies and which are certainly central in the construction of his personality as a teacher.

Moreover, he speaks of this algorithm of Heron’s as a small, fragile object to be handled with care: “There, I tried a little algorithm”, he says ... “This algorithm is really good, but rather delicate”. This way of speaking of a mathematical procedure made us aware of the affective charge he attaches to it.

Summing up, everything seems to point to the fact that his relation to the object ‘square root’ was constituted in the midst of his student past, at a time when the choices of the institution with regard to teaching it were of quite a different nature from those of the present day. One could think that the relation that he constructed with mathematics at that time is a constituent element of his professional identity as a mathematics teacher, that is to say it provides a basis for his professional self.

However, he makes us realise that all the values he created at the time his professional identity was constructed have crumbled, when he speaks of his own methods of calculating in terms of them being “done away with” and “tossed in the bin”.

The tenor of his discourse is consequently lightly depressive, convincing us that these objects of old knowledge were constituted for him as internal foundational psychic objects and that their loss results in a depressive movement which the interview brought back to life. One of his defence mechanisms with respect to this dynamic seems to be to turn the attack outwards and in particular towards those external objects which calculators represent: these, indeed, seem to carry all the ills and the expression of his resentment toward them is very strong, as if the institution had forced them on him as substitute objects with respect to his own lost ones.

He says, for instance, with respect to the extraction method, that it is:

> an important way of thinking [... and] all technical facility is going to disappear completely in the coming years.

One almost has the impression that the very foundations of his relation to mathematical knowledge have crumbled. In fact, it is actually the institution which has suppressed his preferred objects from the syllabus and imposed a certain calculator use on him, but he spares the institution in his discourse and instead concentrates his aggression on calculators. It is these which have become 'the bad object', able to cause the good, internal objects to deteriorate.

Conceivably, this interpretation goes further than the case of Jacques. Indeed, a few studies have shown that mathematics teachers, from junior high school to the university level, resist the integration of calculator usage into mathematics teaching. Perhaps they are displacing onto the calculators the weight of the loss of their preferred objects of fundamental knowledge, which they have had to replace?

We are quite conscious that this interpretation is particularly valid for teachers of Jacques' generation. Moreover, for teachers being trained at the moment, things are reversed given their familiarity with calculators and they are completely surprised to learn of the existence of this 'hand' method of extraction of square roots, just as there is a hand algorithm for carrying out division.

But what emerges through this study of the particular case of the teaching of square root, in conjunction with the choices of the institution, is the fact that teachers can feel assaulted in the foundations of their relation with knowledge by certain institutional choices. The institution is, unawarely, almost 'abusive': in a certain way: it is psychosis-inducing towards them, forcing them almost to dissociate themselves in a double bind:

- *either* they are aligned with themselves and with what has been the basis for their relation to mathematical knowledge and hence they are placed in the wrong *vis-à-vis* the institution with regard to the mathematics it requires them to teach;

- *or* they conform to the requirements of the institution, and thereby they find they are in conflict with themselves at least with their deep, mathematics-teacher self.

In either case, the teachers are constrained to having to contend with a defensive system in order to alleviate the psychic suffering that it has caused them.

The global analysis of the interview resulted in our thinking that for this teacher the status of the interviewer (a university-level researcher) had some important effects. Jacques gives the impression of having wanted to show his best side with respect to his academic knowledge. Moreover, he seems to have experienced the interview situation as an evaluation of his teaching and so believes that he should show his competence with respect to problems of teaching and learning. By means of indicating that the interviewer is interested in a topic where there is nothing to say, or only very little, he tries to reverse roles: his last utterance is a question turned back at the interviewer, as we have already remarked.

The weight of the context of utterance is so strong on this matter that Jacques feels ill at ease. One can imagine that the setting of the interview reproduces for him in miniature the institutional setting which attacks his internal objects. In his initial replies, he negotiates to placate what he feels as threatening and a danger, and to present himself as a subject conforming to the institution. In the heart of the interview, he trivialises the problem while becoming somewhat depressed. At the end, he reverses roles by taking the position of the interviewer, as if a part of him identified with the external aggressor in order to defend himself.

In fact, the internal conflict was inhibited, he wanted to forget it, and the interview situation brought it back to the surface. In addition, his discourse bore witness to his inability to attack the institution which nevertheless has mistreated him directly; he 'preferred' to pay for his submission, that is to say he 'chose' a certain masochistic suffering. The psychic price is undoubtedly less for him than if he were to attack the institution, with its fantasised associated risks.

Recall here that it is from a clinical point of view that we are examining these interviews. From this perspective, we are giving a meaning to the latent discourse of those interviewed, doing so outside any value judgements. We are seeking to understand the psychic processes at work behind the behaviour reported in the discourse. In particular, this method of investigation has a heuristic value: it essentially rests on elaborating the counter-transferential feelings of the researchers with regard to the discourse of the interviewees, a discourse taken in all its expressive extent.

In this instance, the three authors have all had pre-service mathematics training and still teach (or have taught for a long time) this topic. Hence, the words of the teachers which we are analysing find a broad echo in ourselves: it is in ourselves, drawing on this experience, that we are able to let go, to associate with the words of the interviewees and offer interpretative hypotheses.

The analysis of Jacques' interview allowed us to introduce the question of some possible effects of a sort of institutional mistreatment of the relation of mathematics teachers to knowledge. By analysing a second interview, that of Paule, we will find the same problem setting but discover a different means of coping with difficulties arising from institutional changes.

Paule, or confirming institutional choices under the guise of the good of the pupils

Paule is about forty when the interview occurs, a junior high teacher in a very small provincial town. We find the same interview setting as for Jacques (same interviewer and same topic for investigation), but Paule's words reveal that the interview setting is less threatening for her than for Jacques: she even uses the opportunity to elaborate in part on a conflict which otherwise would perhaps not have manifested itself.

In the middle of the interview, one gets the impression of repressed elements coming to light, as the following words suggest:

> But it's true that at the beginning that's the way we used to present the sets of numbers. Now the proof is that I no longer think of it that way any more.

The construction of number systems seems to have served as a core foundation of Paule's relationship to mathematics. She says:

> I've always been fascinated by these means of enlarging sets of numbers and finding properties which extend themselves once again.

However, the new syllabus no longer constructs systems of numbers by successive nesting and, therefore, Paule has been pushed to give up this coherent viewpoint on numbers, which was (as she tells us) the basis of her training:

> I had difficulty with a certain number of things which was how we were trained.

In fact, right from the outset of the interview, all was spoken, the catastrophe was declared:

> There is no more theory of numbers.

And for her, it is clearer than with Jacques: it is the fault of the new syllabus. The institution has constrained her to give up an internal psychic object, one undoubtedly extremely structuring for her professional self. But at what cost? How does she process the suffering entailed by this loss?

For Jacques, one could see depressive feelings come to the surface: for Paule, there is also a depressive tenor throughout the interview, but lighter in tone than Jacques'. Undoubtedly, she has had more success than he has in coping with this loss, especially in valuing her pedagogic function of adapting to the pupils compared with her colleagues who, according to her, could not adapt to the changes in students.

> My colleagues at school are older and they have taught the previous syllabus longer and remain more attached to it.

This is what she emphasises. On the other hand, one has the impression that she is not very invested in her students, even conveying a slightly critical tone when she speaks.

> You finish up by adapting to the kids.

Perhaps her aggression towards the institution is shifted onto the pupils. Having recourse to the idea that these institutional choices are "for the good" of the pupils perhaps prevents her from getting into the double bind we indicated for Jacques: she rationalises institutional choices by denouncing the inability of the pupils to come to this type of understanding.

> It's satisfying for the teacher, but the pupils don't get it at all.

She also relies on the idea that competent individuals have made these choices:

> My way of seeing things is that we have this syllabus, there are people who have reflected on them for us, and we should stick to that. I'm confident.

Patricia, or questioning and the quest for solutions

We see with Patricia another way of reacting to this questioning. Patricia is about forty years old at the time of the interview and she teaches in a junior high school in a very small provincial town in the south of France. Right from the outset of the interview, Patricia expresses her surprise by the question she has been asked, that is the question of the introduction of irrationals in teaching.

> Irrationals? It's true that that's a question that I have never asked myself.

In fact, in these opening sentences one feels a certain shifting around the very notion of irrationality: irrationality does not seem completely separate from non-decimality. [10] Does it arise from her identification with the difficulties of the pupils? She says, in effect, that their first contact with irrationals is in fourth or fifth grade with division when they meet "a number that doesn't stop" (so here it is a question of non-decimal rationals). They never meet the word 'irrational'.

Right from her opening sentences, the word 'number' recurs often. But in her eyes, the pupils do not ask themselves questions about number, they do not see it "existing as a separate mathematical entity". They do not develop in their conception of number: "they never wonder what a number is".

She has the impression that this difficulty is reinforced by the use of formal notation which produces the effect of pupils not knowing that a letter represents a number. The nature of decimals is no longer perceived by the pupils, so for her they are unable to gain access to the structure of numbers integrating decimals.

A formal way of writing prevents them from seeing these decimals are numbers. By contrast with what she tells us pupils cannot perceive, one feels that she holds on tightly to the idea of the structure of numbers and the question of the existence of these mathematical entities:

> They don't ask themselves questions about number; they don't think that such a strong structure lies behind them.

This structure is visibly hers, no doubt coming from her training. One senses a certain distress at not being able to have pupils share in this profound sense of number when teaching the square root.

> From the pupil's side, once Pythagoras has been done in class, it doesn't disturb them much to press the square root button, to find a number and to round it to the nearest tenth or hundredth, depending on what's been asked.

In relation to square root, she insists:

> They can't conceive it's a number, they don't ask themselves questions about number, they haven't even constructed decimals, they don't see them clearly because they round them, they don't have a clue that there is a structure that can be given to numbers, they don't think that underneath there are things to construct.

As the interview goes on, she uses the occasion to expand on this question that she had not asked herself: she takes back into herself the difficulty she had initially projected onto the pupils and questions herself about her teaching:

> I am not sure they will learn about number, I am not sure the conception of number is acquired. For me, the problem is that I just do little bits and pieces, it's too fragmented. I'm not sure if I know how to construct number, I can't manage it.

One can see that this teacher inhabits the notion of number, as if it were a question of an internal object that is extremely structuring for her mathematics teaching self. We might hypothesise that it is one of the determining elements of her relationship with mathematical knowledge. However, as we have already stressed, the syllabus in place at the time of the interview (1985-1995) was such that constructing the irrational numbers was not officially sanctioned in ninth grade. She is anxious about this restriction created by the institution:

> the worry that I think I have even though I don't know the high school syllabus well is the thinking about numbers done at a particular moment [...] meaning should be given back to this number and I don't think the syllabus provides it.

She believes other teachers are less concerned with this question than she is and is not sure that all teachers give meaning to numbers as she would wish. This comment underlines for us the magnitude of this question for her: it is really a 'concern' for her.

It can be seen clearly in the discourse that Patricia finds herself in conflict between her deep conception of the number aspect of square root and what the syllabus does not explicitly require to be taught to pupils. It is as if her internal objects were under attack from the official pronouncements of the institution, and she almost proposes changing the syllabus:

> I'd almost get rid of operations on square roots.

You can see here the strength of her investment in the object of number in the excessiveness of her words:

> We do these operations on square root in the ninth grade and there is no problem there which gives them meaning [...] they should remove the operation side from the ninth grade syllabus.

This interview is exemplary in the way in which a teacher can take hold of the interview setting to elaborate a question on which she has not consciously and systematically reflected before. One can witness the unfolding occurring as the interview develops.

Patricia realises the impediments which the syllabus brings her and at the same time conceives of possible activities which could soften the problem which bothers her:

> I'm just thinking that there is one thing I didn't do with irrational numbers, that I perhaps didn't do enough with rationals which is putting them on lines, axes. [...] I'm realising that putting irrationals on a marked axis is something that I don't do often enough [...] it's true that I've not often made use of the fact that putting them on a line would be one more thing to do.

Her elaboration allows her to realise her desire for her teaching to be less fragmented, to allow her a better place to stand and provide a stronger basis for a more coherent construction:

> the problem with me is that I do little things, it's too chopped up I don't know how to find something that joins it up, how to join up all the little bits that's my problem, I'm afraid of just dredging up little bits of things.

She is particularly aware of the restrictions the syllabus imposes and their influence on her teaching. With regard to $\sqrt{7}$, for instance, she would like to have the pupils realise it is a number, but she explains:

> I think that the numerical aspect is never required in the syllabi [...] I'd like it myself, but I am not sure the syllabi take it into account.

To do this, she thinks of types of tasks like "put a square root between two whole numbers or two decimals", and these thoughts gives rise to a sense of loss with respect to previous syllabi which, according to her, made room for this type of task.

> At one time, with the previous syllabus, it was required, there was quite a lot of work on inequalities which was required with tables of squares, this has been taken away, I think it is no longer done.

This teacher is deeply preoccupied with the numerical aspect of the square root which she finds difficulty with having her pupils construct. All through the interview, she reveals that the current syllabus gives her difficulties with regard to this preoccupation. It seems to us that this questioning is well summarised in the statement she makes practically at the very outset of the interview:

> [In relation to $\sqrt{17}$] I would like it to be a number. I would like them to know it's a number.

It is very important for her but is also something to be shared with the pupils. Moreover, this is where she ends too in the final moments of the interview:

> Teaching the square root in the end is often a failure with my pupils [...] as this is the case, it seems to me that what is indispensable is to construct the meaning of the square root.

Claude, or the transmission of traumas

With the fourth interview, that of Claude a forty-two-year-old teacher in a junior high school in a medium-sized town in the suburbs, we embark on a new aspect of the problem. In effect, if with each interviewee in this enquiry we find the question of the loss of foundational objects for their relation to mathematical knowledge, with Claude we discover that the problem begins considerably prior to the present-day changes proposed by the institution and that the injuries are perhaps a trace of very old traumas that the institution allows teachers to endure without providing them with sufficient means to cope.

At the outset, notice that in this interview the interviewer had to increase the number of prompts, because Claude expressed himself in a succinct and limited way. He did not take advantage of the opportunity, as Paule or Patricia were able to do, to carry out his own reflection.

Claude's discourse made us feel that this teacher was not strongly invested in his pedagogic work, and that he was little caught up in the interview emotionally. His emotions were barely perceptible, his manner of speaking seemed to mirror what he led us to understand about his somewhat mechanical behaviour in class.

He sees pupils as leaky reservoirs, to be repeatedly filled, all the while feeling there is no real solution:

> You must realise that nevertheless you have more than half the pupils who are only functioning with conditioned reflexes.

On the pedagogic level, he seems a little *blasé*. To the question "What do you do for your pupils?", he replies:

> It's repetition, I've never found a solution. No teacher gave me a recipe [...] you explain to them, you succeed in having them accept it, but when you take it up again a fortnight later half of them get it wrong [...] They retain it because you beat them up all year so that they do it.

He very commonly employs a vocabulary of warfare, often against the pupils:

> you beat them up all year [...] you have to hit them so they maintain the exact way of writing radicals, you must force them.

He wants to block escalating errors, to avoid "errors that they repeat afterwards".

This aggressive way of speaking is also used against mathematical knowledge: Claude wages the "war of approximate values", which is "dreadful", "you hit up against this all year" and "there are plenty of traps there".

For him, mathematical knowledge is about "exact values", "exact writing", results must be put:

> in the exact form [...] it's the governing idea in grade nine. You have to force them to work with the exact values.

Once again, it is war:

> You have to struggle so they write exactly with radicals.

The word ‘exact’ appears at least six times in about ten lines at the beginning of the interview and it returns throughout the discourse. In response to a question about a technical exercise on getting rid of radicals in the denominator of an expression, he remarks:

> it’s not a central goal. Another goal is the notion of exact value.

At another moment, he brandishes the institutional flag:

> In all the problems of the *brevet* [11], they always require results in the exact form.

For him, the notion of exact value is confused with the notion of exact result and all of a sudden it leads to chasing after errors:

> you should either keep fractions or radicals to produce exact results in which no errors are introduced any more.

If we bring together this obsession with the exact value, the fact that for Claude it is a matter of chasing after approximate values with the his use of war vocabulary, it seems to us that the war in question is a sort of crusade to rid mathematical knowledge of its impurities. Knowledge must be disinfected, you must have quite clear frameworks. So he insists on “perfect squares” to give pupils a proper framework.

One could say that mathematics represents for him a universe of purity and that he has to go hunting for the impurities introduced by the pupils. But, unfortunately, this universe is somewhat perturbed by these irrational numbers, of which he says:

> you have to make them acknowledge that those roots are not normal numbers, numbers which never come out exactly.

Moreover, he calls these numbers ‘special’ at the beginning of the interview and ‘bizarre’ at the end. This mathematical knowledge which he would like to be pure is already contaminated before the pupils mix it up. He would like to be the spokesperson of an exact science, but these ‘bizarre’ numbers get in the way.

We were unable to prevent ourselves from feeling that everything transpired as if Claude were reliving the trauma of the high mathematician (albeit in a minor key of course), that moment when the radiant heavens of the Pythagoreans collapsed following the discovery of irrationals. The historian Szabo (1977) indeed speaks of “scandal” and “criminal treason against the Pythagorean doctrine” with regard to divulging mathematical irrationality.

According to several sources, the one responsible for this treason:

> would have been hurled [...] into the sea for impiety. (p. **xx**)

From the same point of view, it also seems to be the case that the Greek word currently translated as 'irrational' corresponds instead at the etymological level to 'unsayable' (*a-logon*). The terms Claude himself uses – 'non-normal numbers', 'numbers which never come out exactly', 'special numbers', 'bizarre numbers' – call to mind the terms used by Simon Stevin in his 1634 *Treatise on Incommensurable Magnitudes* to criticise certain writers:

> It is a vulgar thing in arithmetic authors to deal with numbers like √8 and similar, and which they call absurd, irrational, irregular, deaf, etc. (p. **xx**)

One can see through these elements that this discovery did indeed act as a trauma for Greek mathematicians and that its effects have traversed history, since traces are still perceptible today. When Claude says:

> The exact value has something of the unreal for pupils, it doesn't exist in reality, there is something quite deep in it over which we don't linger.

one can but wonder if this statement does not reflect his own doubts on the mathematical reality of irrational numbers. More generally, this statement perhaps gives us a sense of his conception of mathematical objects. It seems that it rests on a material conception and that it has not really evolved towards a conception of mathematical objects as mental constructions.

> They have trouble thinking of √17 as a number representing a length.

What is more, he thinks:

> that they manipulate them without really knowing what there is behind.

Might not this difficulty with knowledge projected onto the pupils reflect his own (current or past), brought back to life through contact with the pupils?

One can wonder to what extent he is transmitting his own traumatic relation to knowledge to the pupils – considering the fact that he emphasises that no one gave him recipes (and that the pupils would like recipes that he has not been able to give them) – which he himself has not really got over. The institution has not given him the means and one could even say that this institutional silence pushes him towards a form of regression.

We are witnessing the effects of the after-shock of the traumas of the high mathematician in relation to teacher knowledge and the transmission which they carry out. To what extent are these effects due to or rather reinforced by the institution not taking the crises of the high mathematician into account?

One can also surmise that the teachers who have repeatedly had to accompany the pupils as they overcome this trauma are lead to relive and re-accustom themselves to its traces through contact with the pupils. That comes into play in two senses: they transmit their own unthought thoughts and doubts about their relation to knowledge and, reciprocally, the life of the pupils revives their encounters with these traumas each year (Blanchard-Laville, 1998).

Linked traumas in the genealogy of knowledge

In summary, we wish to emphasise that for the first while, we were particularly attuned to expressions of nostalgic sentiments with respect to teacher training in the discourse of these teachers and even their regrets about the mathematical knowledge they had encountered during their studies, which were foundational with respect to their identity as mathematics teachers.

Recall that these interviews concerned the question of introducing the square root in junior high or high school, an introduction which takes place in the context of access to the real numbers. However, without a doubt for mathematics teachers, the *real numbers* comprise a central object of mathematical knowledge that they have chosen to teach. This object can even be considered as one of the most significant on which many theoretical constructions rely. It is almost an axis which structures the advance of mathematical knowledge, a sort of spinal column which supports the edifice of algebra and analysis. From this point of view, these objects, encountered by students who will be future mathematics teachers, comprise a sort of core initiation of their training.

At the level of the history of mathematics, the development of the notion of number has provoked several grave crises, from the Greek discovery of irrationality to the elaboration of the formal constructions of the set of real numbers toward the end of the nineteenth century. It is precisely because the foundations of analysis were not sufficiently 'solid' or too dependent on an intuitive geometry that mathematicians toward the end of the nineteenth century sought to come up with more rigorous constructions for the set of real numbers.

On the heels of this, mathematicians sought to master the notion of infinity and this attempt triggered a new crisis of knowledge which gave rise to the discovery of what are called *transfinite* numbers: these numbers are in some way the measures of different infinite sets which allow sets to be compared from an order perspective and upon which an arithmetic can be constructed.

From our point of view, all these crises in mathematical knowledge can be interpreted as traumas, in the sense that the foundations of the edifice of knowledge are shaken by forces that the system cannot contain as it stands. Indeed, contradictions and paradoxes multiply and provoke thoughts that even present-day knowledge cannot absorb. A few of the new discoveries subsequently allow the crisis to be transcended, but in our sense, the traumatic effect will remain inscribed in the knowledge itself.

Traces of these traumas exist, for instance, in the names for the objects themselves ('irrational' since the Greeks, 'real' since the seventeenth century, as right from the start Descartes suggested this term in opposition to 'imaginary' [12]), which attests to the residues of the traumatic relationship mathematicians have to these objects. Most of the time, the traces have been effectively erased. But in a recurring manner, the crises resurge, undermining once more the mastery acquired up to that point. On each occasion, the new equilibrium does not prevent other paradoxes emerging into the light of day, engendered by the new complexity.

Each student of mathematics (and potential mathematics teacher) is going to be confronted unconsciously in the process of their training with latent traces of this history. In general, knowledge is presented to them in a smoothed out and successful version, a form which has erased and absorbed the historical crises which have taken place and which, in a certain way, denies the time that has been necessary to overcome them. What is more, one could make the hypothesis that the student recipients of this relatively sanitised but not scar-free knowledge are not able to overcome such crises without unconsciously feeling the traumatic effects of this history.

The traumas of the high mathematicians have repercussions at the level of school mathematical knowledge by means of the inevitable didactic transposition (Chevallard, 1985) which the teaching institution brings into being. Moreover, the institution bears witness to its difficulty in managing the crises induced by the high mathematicians both by the succession of teaching strategies proposed throughout all the syllabus changes and in certain cases by the 'voids' it creates in the curricula.

It is especially visible and significant in the case we are studying here, the introduction of the real numbers. We know that the so-called 'modern mathematics' reform was promoted in the 1960s by a team of mathematicians whose project was to reduce the gap between the mathematical knowledge taught and research knowledge: a team certain of whose members took part in the Bourbaki adventure of reconstructing the entire mathematical edifice of the time.

Note that the essential characteristics of this restructuring were to a certain extent bound up with making an attempt to fill in knowledge all the while ensuring the perfect coherence of the set notion within the framework of an axiomatic rigour which would be beyond reproach. [13] One wonders whether it was not a question for them of trying to master different pasts traumas as well as those yet to come.

The mathematics teaching reform project in junior high and high school was established by the institution which appropriated it, but could only maintain it for a few years and subsequently took part in what is called the 'shutting down of the reform' in 1978. This created a void in the knowledge to be taught, producing a void at the level of official instructions and syllabi. [14]

This new institutional relation to the object of real number is in response to the institutional crisis prompted by the reform. It is as if the institution in its turn had difficulties in negotiating the trauma of the high mathematician with regard to the irrationals (Bronner, 1997a, b).

Can one actually speak of institutional trauma? For quite a long time (the classical period 1850-1950), the system was stable. At a given moment, by means of a mathematical advance and the thinking of mathematicians who wanted to bring this new knowledge to it, the system was to some extent assaulted, the old bases crumbled and it was unable to cope with the surfeit brought with it. This surfeit of agitation could not be absorbed – in this sense, we think, one can actually speak of institutional trauma.

For the students of mathematics, future mathematicians or teachers of mathematics, one wonders if the teaching they receive is not like an unconscious undertaking of denial of history and its crises and also, undoubtedly, a denial of living subjects who created this history. All the objects which were problematic throughout history no longer appear as such: they are routinised and now show up at the heart of mathematics as ordinary objects like any other. They are put to work by logical construction, detemporalised and impersonal. We remain mute – or almost so – about the traumas, somewhat like children who have suffered violence about which they remain silent, hoping that "it will take care of itself".

When these students become mathematics teachers, it is likely that the effects of all these phenomena will surface. As subjects in the teaching institution, they are going to have to adapt to the choices it requires of them when confronted with obstacles which mathematical knowledge has already had to face. In their turn, at their own individual level, they are going to have to find ways of contending with institutional requirements in conjunction with their own relation to mathematical knowledge and so come to compromises which are professionally tenable. And then the pupils will be touched, by the last vestiges of these traumatic effects, though in a manner dependent on how their teachers have contended with these difficulties. Finally, the teachers will have partially or differently to relive these traumas that the pupils themselves experience.

In this analysis of four interviews, we have been able to sketch how these denials resonate down to the deep core identity of the teachers, which has led us to believe that in certain cases the teachers experience a form of institutional mistreatment.

And this seems to us to be even stronger in the current period as the **xxx** [transpositifs] choices which are in effect magnify the effects of the void: because of this, the teachers can no longer acknowledge at all in their teaching something which could have formed a basis for their professional selves as mathematics teachers.

This research has had specific bearing on the teaching of mathematical knowledge and even on certain specific points of that teaching at the current time. It seems to us that if our hypotheses have some relevance, they should give rise to broader means of understanding which could relate to other particular points of mathematical knowledge in connection to the choices of the teaching institution, and also no doubt to other types of knowledge taught in schools than mathematics.

Acknowledgement

This article was translated into English by David Pimm, with assistance from Nathalie Sinclair. It first appeared in French in Mosconi, N., Beillerot, J. and Blanchard-Laville, C. (eds) (2000) *Formes et Formation du Rapport au Savoir*, Paris, Éditions l'Harmattan.

Notes

[1] *Translator's note*: The French term '*le(s) programme(s)*' refers to a compulsory, nationally specified, grade-specific mathematics course ('le cours'). In terms of the extent of elaboration, its meaning lies somewhat between the English notions of 'syllabus' and 'curriculum'. Because its focus is primarily on content rather than pedagogy (though this balance is currently shifting), we have chosen to use the term 'syllabus' almost always

throughout (even though technically there is one *programme* per school subject per grade level), except where the authors specifically choose the term '*le curriculum*'.

Junior high school (*le collège*) is from grades six to nine inclusive.

[2] It is in grade 8 in junior high school that pupils officially meet for the first time the object 'square root'. The square root of a positive number *a* is defined as the positive number whose square is *a*.

[3] CREF stands for the xxx.

[4] This change, promoted by a commission of mathematicians, was put into place in 1968, then abruptly halted in 1978 as a result of teacher pressure. It was intended to reduce the distance between discoveries having to do with the main mathematical structures emerging a century before and present-day teaching. It was thought at the time that an early introduction to these structures would facilitate the learning of mathematics. This idea was reinforced by the fact that certain individuals had seen links between these mathematical structures and the cognitive structures of the developing child elaborated in the work of Piaget.

[5] *Translators note*: The reference of 'suburban' (*banlieue*) in France has many characteristics of 'inner-city' in other Western countries, in terms of pupil background and school conditions. Suburban schools often offer some of the most difficult teaching settings. Frequently, the most tranquil, academic, 'bourgeois' schools are to be found in the centres of medium to large towns and cities.

[6] For instance, roots of non-perfect-square whole numbers, such as √13.

[7] Thales' theorem is a sort of symbol (as is Pythagoras' theorem) of junior high school geometry teaching.

[8] The radical is a symbolic signifier, √, of the square root.

[9] Heron's algorithm is a calculating method for approximating a square root, much older and more effective than 'the method of extraction'. Traces of it can be found with the Babylonians. Starting from an approximate fractional value *a* to the square root of *A*, this method consists in obtaining a better fractional approximation still by calculating the quantity $1/2(a + A/a)$.

[10] A decimal is a particular case of a rational number and there are non-decimal rationals such as 1/3 (which has an infinite decimal expansion). These different characteristics – irrationality and non-decimality – relate to different notions and therefore should be distinguished.

[11] The *brevet* is the national examination at the end of junior high school, namely the end of grade nine.

[12] Descartes' (1637/1984) *Geometry.*

[13] This restructuring goes back to Dedekind (1872) who shows that the set of real numbers obtained by his construction cannot be ‘extended’ and to Cantor (1872) who for himself emphasises the ‘completeness’ characteristic. This was pursued by Hilbert (1899).

[14] We could almost speak of manic defences on the part of the mathematicians with respect to the traumas, and the return of the real when the institution and its pupils get involved.

References

Beillerot, J, Blanchard-Laville, C. and Mosconi, N. (eds) (1996) *Pour une Clinique du Rapport au Savoir*, Paris, Éditions l’Harmattan.

Blanchard-Laville, C. (1998) ‘L’enseignement et la transmission dans l’espace psychique de la classe’, *Recherches en Didactique des Mathématiques* **17**.

Blanchard-Laville, C. (1999) ‘Espace de co-pensée et co-disciplinairié en sciences de l’éducation’, in *Approches co-disciplinaires des pratiques enseignantes dans leurs rapports aux apprentissages des élèves*, Rapport de la recherche commanditée par le CNCRE (Comité National de Coordination de la Recherche en Éducation).

Bronner, A. (1997a) *Étude Didactique des Nombres Réels, Idécimalité et Racine Carrée*, Thèse, Grenoble, Université Joseph Fourier.

Bronner, A. (1997b) ‘Les rapports d’enseignants de Troisième et de Seconde aux objets <<nombre réel’>> et <<racine carrée>>’, *Recherches en Didactique des Mathématiques* **17**.

Chevallard, Y. (1985) *La Transposition Didactique*, Grenoble, La Pensée Sauvage.

Chevallard, Y. (1988-89) ‘Le concept de rapport au savoir’*, Séminaire de Didactique des Mathématiques et de l'Informatique*, Grenoble, Université Joseph Fourier.

Descartes, R. (1637/1984) *La Géométrie*, AREFPPI.

Kaës, R. (ed.) (1993) *Transmission de la Vie Psychique entre Générations*, Inconscient et culture, Paris, Dunod.

Stevin, S. (1634) *Treatise on Incommensurable Magnitudes*.

Szabo, A. (1977) *Les Débuts des Mathématiques Grecques*, Paris, VRIN.

Tisseron, S. (ed.) (1995) *Le Psychisme à l’Épreuve des Générations: Clinique du Fantôme*, Inconscient et culture, Paris, Dunod.